The Minimum Difference in Crossing Number between a Graph and its Cone

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Introduction

A graph is (relatively formally) defined as a mathematical object consisting of a set of both points, known as the vertices of the graph, and the (possibly curved) lines connecting them, known as the edges of that graph. When we consider a graph, we usually do not care about the exact layout of the vertices and the edges, in the sense that their precise location doesn't matter; the only fundamental features of a graph are which edges and which vertices it contains. That is to say, we may rearrange a graph G by moving the vertices and redrawing the edges so that the same vertices are connected by the same edges, and if we do so we merely obtain a different drawing D of the graph G.

A drawing is termed 'nice' if it satisfies three properties: no edge ever crosses itself, no two edges adjacent to the same vertex cross each other, and no two edges cross more than once, and every graph possesses a nice drawing. We're interested in drawing a graph with the minimum number of edges crossing. For example, observe that the graph which consists of five vertices and one edge between every pair of vertices, which we may call K_5 or the complete graph of five vertices, cannot be redrawn so that no pair of edges cross each other. However, we may draw it so that only one pair of edges cross. (Figure 1) We define a quantity called the crossing number, defined for any given graph, which is the minimum number of crossings with which the graph may be drawn. Thus we may say that the crossing number of K_5 is equal to 1, or $Cr(K_5) = 1$. The crossing number for any finite graph is a well-defined value, which can in theory be calculated, a task made easier by the fact that any drawing of a graph with the fewest number of crossings is always nice.

We chose to investigate a specific question in the field: given an arbitrary graph with crossing number k, what is the minimum crossing number of its cone? Here, the 'cone' of a graph G is the graph C formed by the addition of a new vertex, called the 'apex vertex' or a, and new edges from a to every other vertex in G. (See Figure 2 for an example of a graph's cone.) Previous work has resolved this question only for very small values of k, up to at most $k = 5$. We therefore investigated the behavior of cones of graphs with $k = 6$ and $k = 7$ to build upon this previous work.

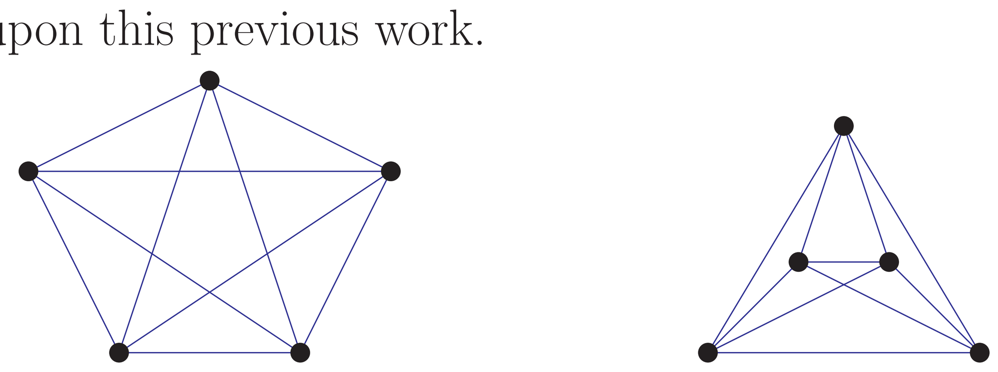

Figure 1: The complete graph K_5, drawn both conventionally and with only one crossing

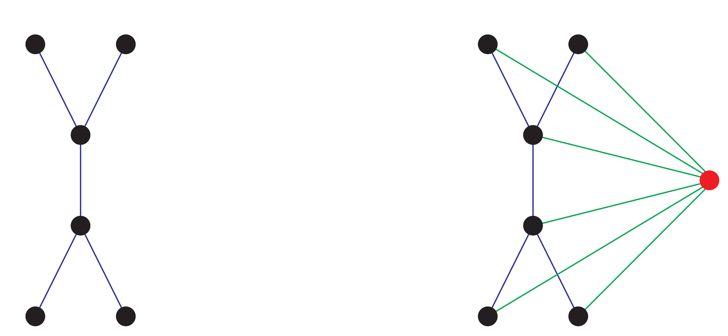

Figure 2: A graph and its cone

Methods

Recalling that we defined the graph G so that $Cr(G) = k$, we observe that $Cr(C) \geq k$. We therefore may introduce a nonnegative integer t, such that $Cr(C) = k + t$. Our task then falls to finding the minimum value that t may take, as a function $f(k)$ of k. This may be accomplished by two methods in concert:

1. We may work towards finding a lower bound for $f(k)$. Finding lower bounds for $f(k)$ generally works by demonstrating that if t were lower than some value, this would lead to being able to redraw G with fewer than k crossings, and therefore deriving a contradiction. Since $f(k)$ is the minimum value t is permitted to take, a lower bound for t is a lower bound for $f(k)$.
2. We may work towards finding an upper bound for $f(k)$. As $f(k)$ is the minimum for t, an example of a graph which results in a specific t for a given k gives an upper bound on $f(k)$. Previous work [1] on this topic has given some simple examples of graphs with crossing numbers ranging from 1 to 5, with the minimum number of crossings in their cones. These examples are found in Figure 3. The case of $k = 0$ is trivial.

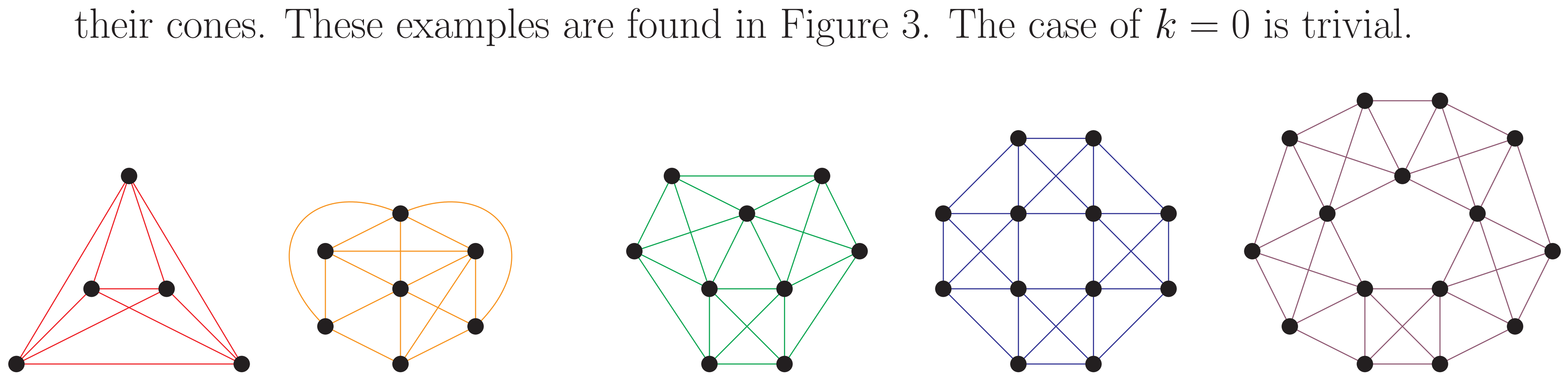

Figure 3: From left to right: $k = 1, 2, 3, 4, 5$. Colored for aesthetics.

Results

We considered the cases for both $k = 6$ and $k = 7$. While the proof remains unfinished, we did find two promising candidate graphs, and probable paths to a complete proof for each candidate graph. One particularly important lemma involves the 'nonplanar neighbors', which are the set of vertices (drawn in grey in the figures below) not found on the outside of the graph. If we say, for a graph G, that D is the drawing of G which appears as part of its cone C when C is drawn with as few crossings as possible, then the lemma says that D must have at most t nonplanar neighbors. This lemma helped enumerate the possible categories of graphs, and provide limits on what they could look like.

For the case where $k = 6$, we were able to find a graph that could be drawn with six crossings that led to a cone which could be drawn with only 11. In this case, it would suffice to prove that the crossing number of the graph is 6.

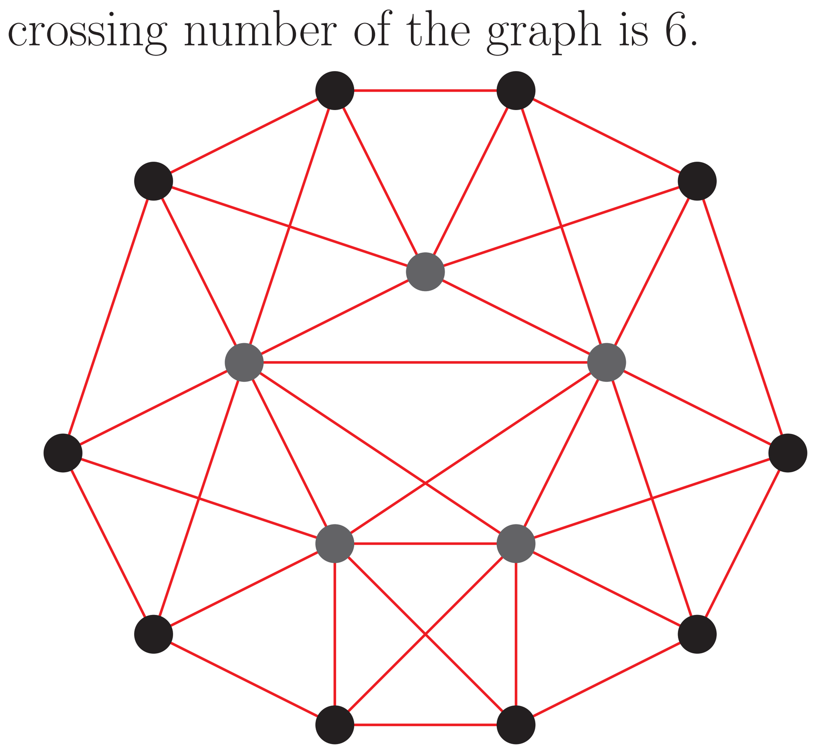

Figure 4: The candidate graph for $k = 6$.

For the case where $k = 7$, we were able to find a graph that could be drawn with 7 crossings, whose cone could be drawn with 13. In this case, it would suffice to show that the graph's crossing number is 7, and that a graph with crossing number 7 cannot have a cone with only 5 more crossings.

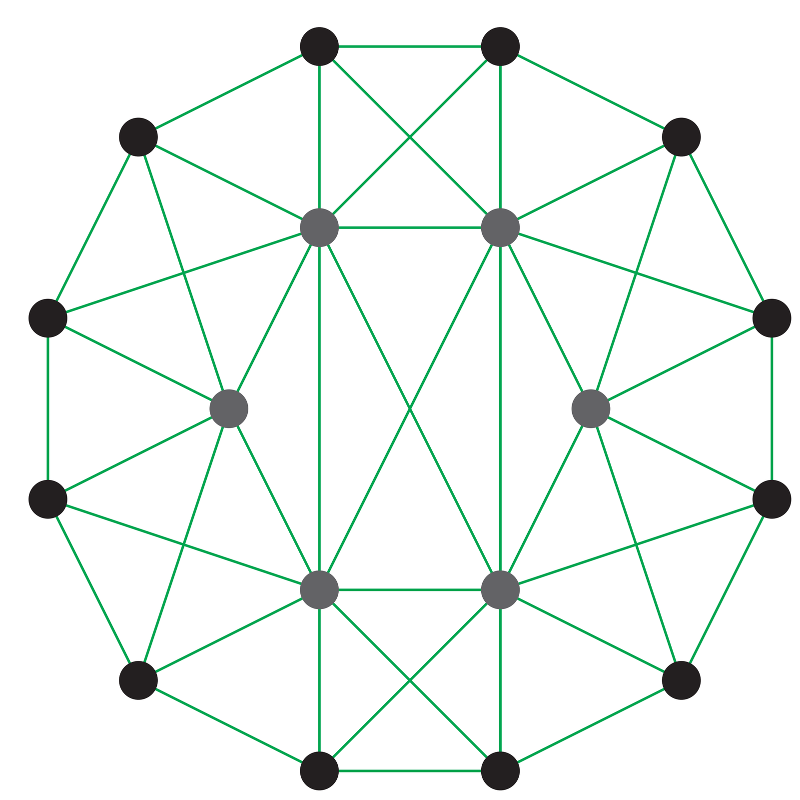

Figure 5: The candidate graph for $k = 7$.

Very conveniently, in both these cases, just as in the cases for $k = 1, 2, 3, 4, 5$, there exists an optimal drawing for G which is also part of the optimal drawing for C, so we don't need to do any explicit redrawing during the course of the proof.

Motivations and Context

The origins of this question lies in the intersection of the study of a quantity known as the chromatic number and the crossing number, for a given graph. The general relation between these is given by the Albertson Conjecture, for any arbitrary graph. Therefore, a better understanding of the relation between the crossing number of a graph and of its cone helps to understand and shed light on the Albertson Conjecture, which itself gives us more tools with which we might understand the crossing number or the chromatic number of an arbitrary graph - questions we do not have many tools to deal with, at the moment. And such questions may be important and relevant in any topic that deals with two-dimensional arrangement of things that we would like to cross as little as possible or to categorize with as few categories as possible, such as the construction of circuit boards.

References

[1] Alfaro, CA *et al.* *SIAM J. Discrete Mathematics*, 32(7):2080–2093 (2018).

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