The Levi-Civita Transform in Two-Body and Three-Body Systems

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Background

The gravitational N-Body problem is a problem in physics to predict the motions of N particles interacting gravitationally. For $N \leq 2$, there exist exact solutions that is easy to compute for all time, given initial conditions. However, for $N > 2$, such solutions do not exist. For $N > 2$, and specifically $N = 3$, we need to use computational methods to study the evolution of the system.

Traditional methods of modeling the motions of the gravitating bodies involve calculating the change in velocity and position for each particle, and stepping through time, as determined by the equation for the gravitational force.

$$\vec{F}_i = \sum_{j \neq i} \frac{G m_i m_j}{r_{ij}^3} \vec{r}_{ij}$$

Traditional numerical methods involve calculating the change in position and velocity, and updating the position and velocity. These methods often involve taking small steps in time, so that the actual change in x and v can be accurately modeled. At long distances, relatively large time steps can be taken, as v changes very little. At close range, small steps must be taken, which significantly increases the time to run a simulation.

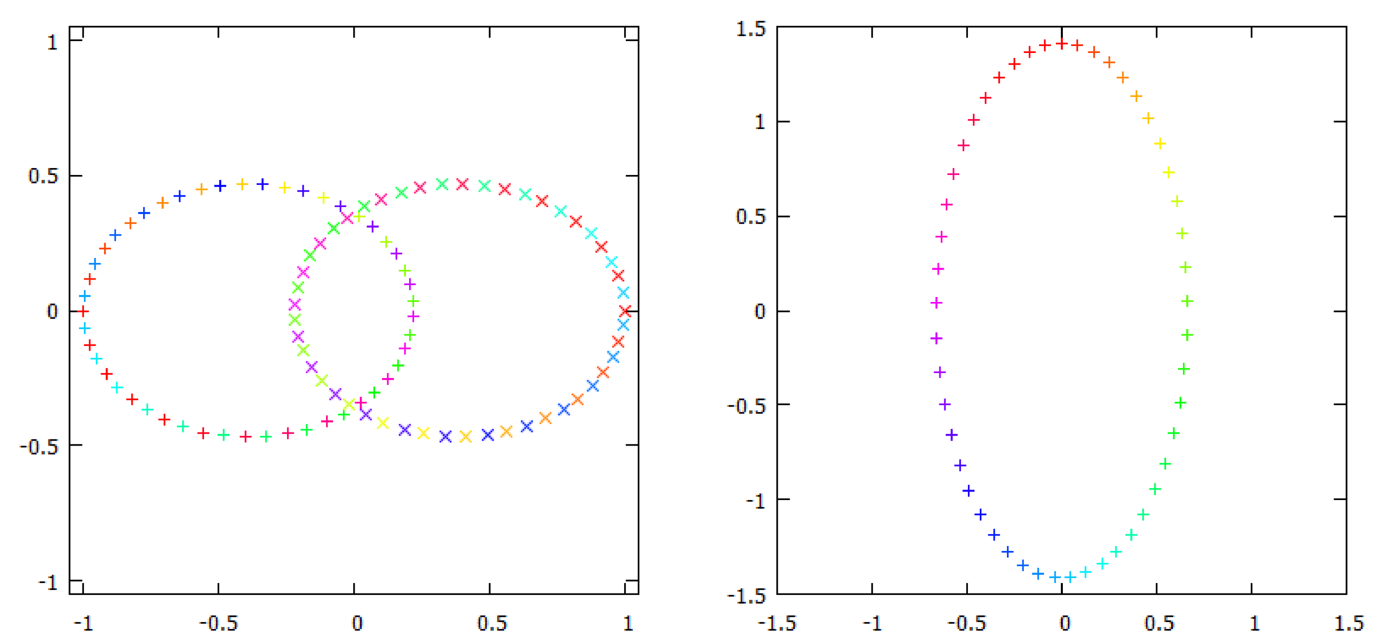
A problem arises when two bodies collide, or get very close to colliding. The gravitational force has no bound, and thus as the points get closer, the acceleration and velocity of each particle approach infinity, which is impossible to model computationally using standard methods.

The Levi-Civita transform is a 2-D regularization that alters time and space, allowing close encounters, or collisions, to be efficiently modeled. The Levi-Civita transform works by shifting coordinates, and then scaling time so that the real time runs slower at collision. Finally, the measurement of energy is shifted so that the energy is zero along the trajectory of the system. This transformations turns the non-physical collision point where division by zero occurs into an actual point, which can be integrated past like any other, while also making close encounters easier to model.

Methods

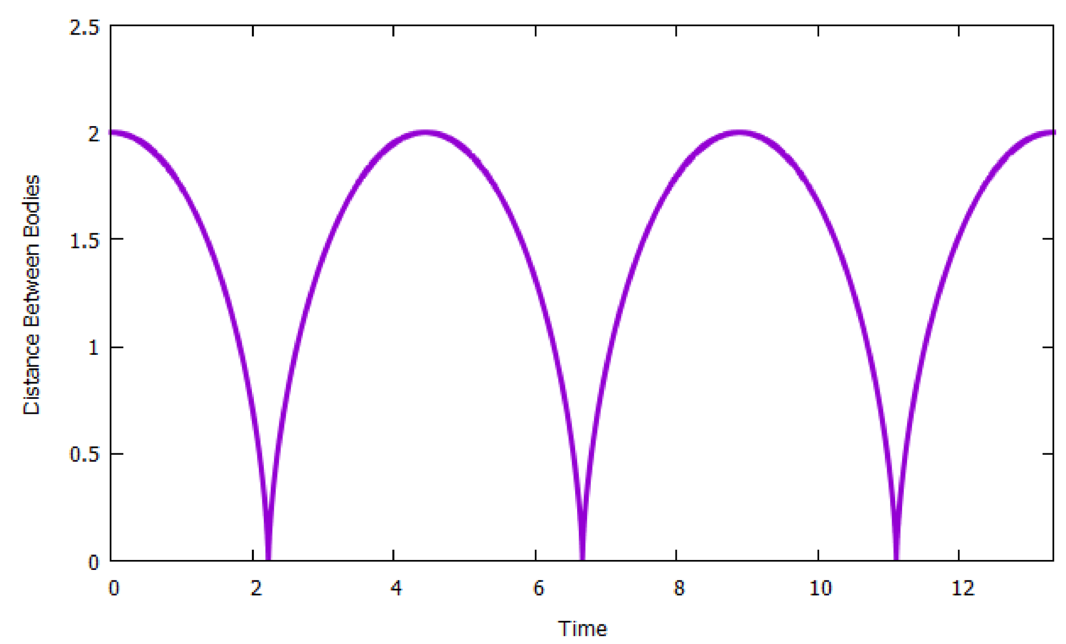
We implemented The Levi-Civita Transform in two-body and three-body systems. The first step in the transform is to treat the relative position and velocity of the bodies, x and v, as coordinates in the complex plane, associating the y coordinate with the imaginary component. We then determine the coordinates in the Levi-Civita Space, and the current time scaling. We then integrate through the Levi-Civita space, by updating the Levi-Civita coordinates, and the real elapsed time. We swap back to regular coordinates when a final outcome is needed.

Two Body Results

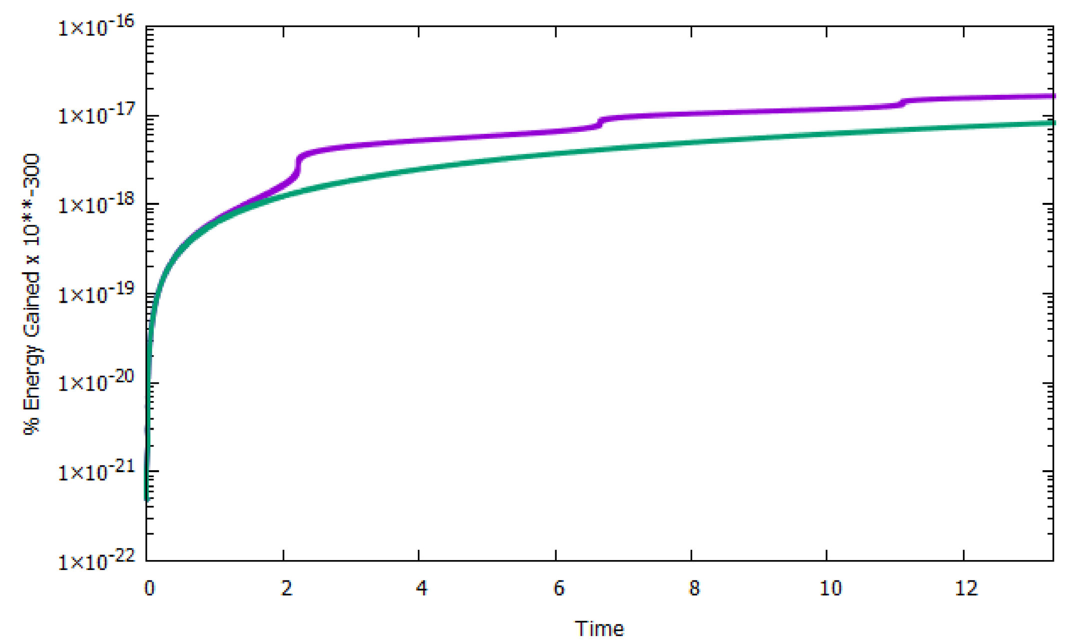


Each of these plots are the same plot of the motions of two bodies orbiting each other. On the left the real coordinates, and on the right are the Levi-Civita coordinates. Points of the same color occur at the same time.

One of the interesting facts about the Levi-Civita transform is that it changes the angles at the origin. Note in the above plot that each body completes the same orbit twice in the same time it takes the point in Levi-Civita coordinates to make one orbit.

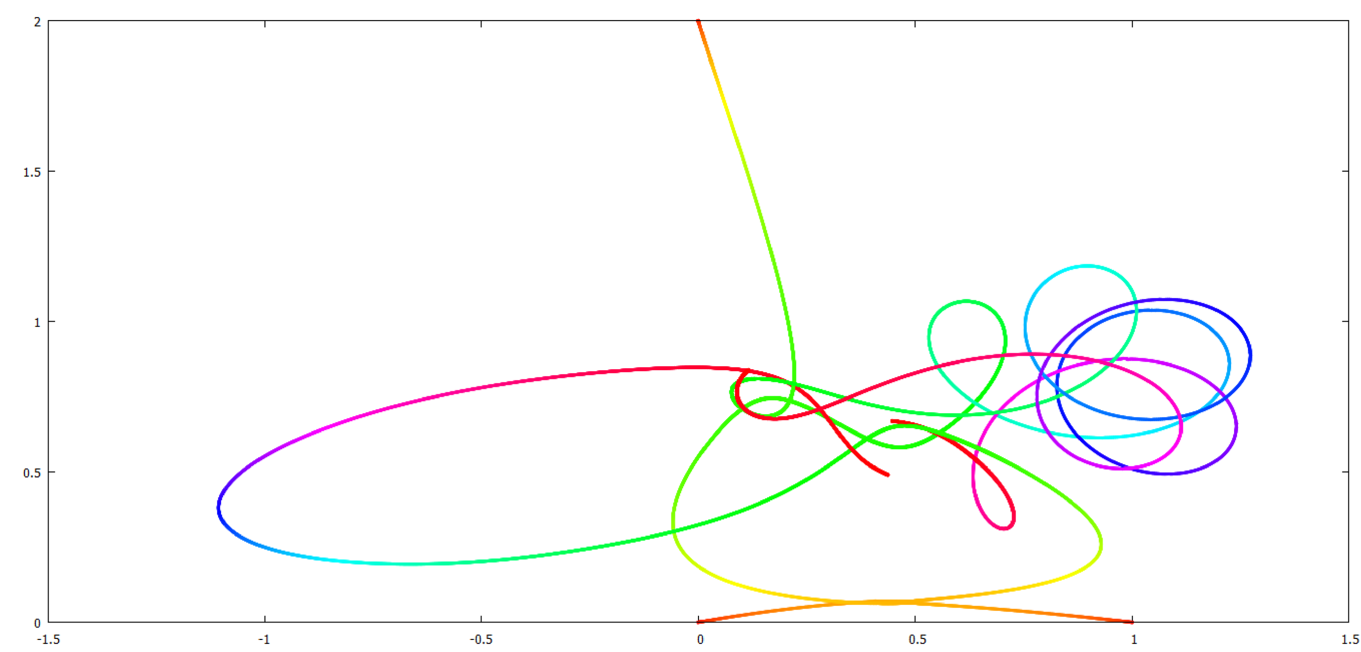


The distance between two bodies during multiple collisions.



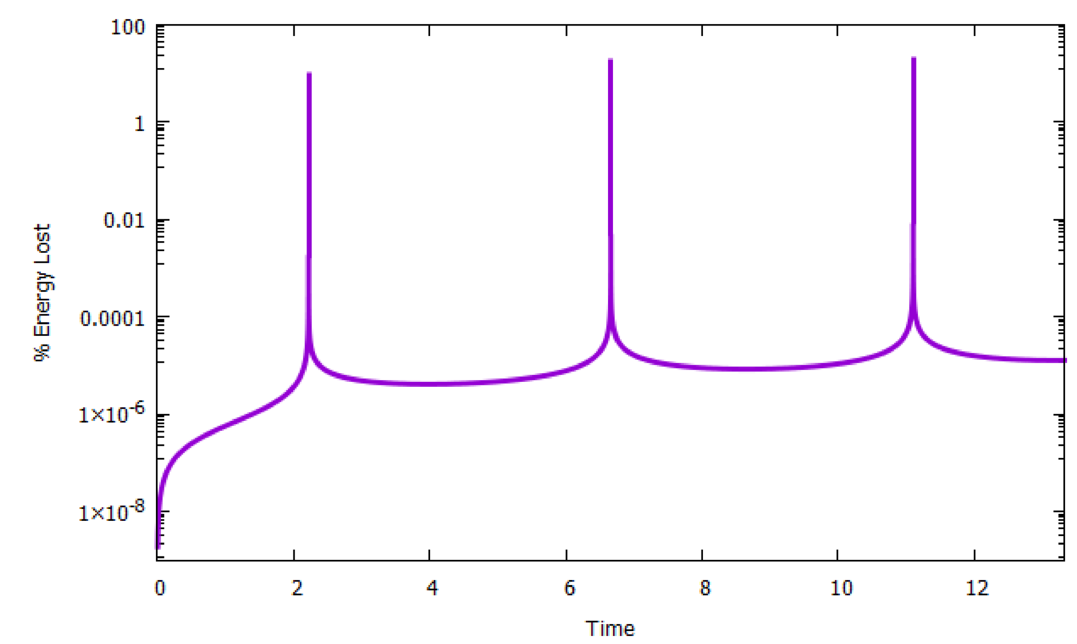
Here we have the percent of initial energy the system has gained. The top line corresponds to two bodies colliding, while in the lower line, the same bodies orbit each other. Note that energy is well conserved.

Three Body Results



The positions of three bodies motions as a function of time.

We successfully implemented the Levi-Civita Transform in three-body systems. Above, we see the paths of three interacting bodies. Points of the same color on different trajectories happened at the same time.



Percent energy the system has lost while two bodies repeatedly collide, while the other is far away. Note that the percent lost spikes at collision, and then returns to a normal value soon after.

Future Work

1. We plan to use this code to examine the collisions of binary-single star collisions, and in particular collisions that result in otherwise difficult to model interactions, such as when a head on collision occurs.
2. We plan to look at how regularizations are made, and attempt to regularize the collisions between between atoms.

References

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[2] Alessandra Celletti.
Basics of regularization theory.
Chaotic Worlds: From Order to Disorder in Gravitational N-Body Dynamical Systems, 2006.